The Superfluid State of a Bose Liquid as a Superposition of a Suppressed Bose-Eistein Condensate and an Intensive Pair Coherent Condensate

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Abstract

A self-consistent model of the superfluid (SF) state of a Bose liquid with strong interaction between bosons is considered, in which at $T = 0$, along with a weak single-particle Bose-Einstein condensate (BEC), there exists an intensive pair coherent condensate (PCC) of bosons, analogous to the Cooper pair condensate of fermions. Such a PCC emerges due to an effective attraction between bosons in some regions of momentum space, which results from an oscillating sign-changing momentum dependence of the Fourier component $V(p)$ of the interaction potentials $U(r)$ with the inflection points in the radial dependence. The collective many-body effects of renormalization ("screening") of the initial interaction, which are described by the bosonic polarization operator $\Pi(\mathbf{p}, \omega)$, lead to a suppression of the repulsion $[V(p) > 0]$ and an enhancement of the effective attraction $[V(p) < 0]$ in the respective domains of nonzero momentum transfer, due to the negative sign of the real part of $\Pi(\mathbf{p}, \omega)$ on the "mass shell" $\omega = E(p)$. The ratio of the BEC density n_0 to the total particle density n of the Bose liquid is used as a small parameter of the model, $n_0/n \ll 1$, unlike in the Bogolyubov theory of a quasi-ideal Bose gas, in which the small parameter is the ratio of the number of supracondensate excitations to the number of particles in an intensive BEC, $(n - n_0)/n_0 \ll 1$. A closed system of nonlinear integral equations for the normal $\tilde{\Sigma}_{11}(\mathbf{p}, \omega)$ and anomalous $\tilde{\Sigma}_{12}(\mathbf{p}, \omega)$ self-energy parts is obtained with account for the terms of first order in the BEC density. A renormalized perturbation theory is used, which is built on combined hydrodynamic (at $\mathbf{p} \to 0$) and field (at $\mathbf{p} \neq 0$) variables with analytic functions $\tilde{\Sigma}_{ij}(\mathbf{p}, \epsilon)$ at $\mathbf{p} \to 0$ and $\epsilon \to 0$ and a nonzero SF order parameter $\tilde{\Sigma}_{12}(0, 0) \neq 0$, proportional to the density ρ_s of the SF component which is a superposition of the BEC and PCC. In the framework of the "soft spheres" model with the single fitting parameter—the value of the repulsion potential at $r = 0$, a theoretical quasiparticle spectrum $E(p)$ is obtained, which is in good accordance with the experimental spectrum $E_{\exp}(p)$ of elementary excitations in superfluid ^{4}He.

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1 Introduction

An ab initio computation of the spectrum of elementary excitations in the superfluid (SF) ^{4}He Bose liquid remains an actual problem nowadays, despite certain recent successes in that direction, like an excellent agreement with experimental data in the region of the roton minimum obtained by the Monte Carlo method making use of the so-called "shadow wave function" [1] and by the corellation basic function method [2] employing modern interatomic potentials for ^{4}He [3]–[5]. At the same time, a microscopic field perturbation theory [6]-[8] calculation of the long-wave phonon part of the spectrum $E(p) \simeq c_1 p$, where c_1 is the speed of first (hydrodynamic) sound in liquid ^{4}He, faces principal difficulties. This is due to the fact that nonrenormalized perturbation theory gives rise to infrared divergencies and nonanalyticities at $p \to 0$ and $\epsilon \to 0$ [9]–[12], which can be cured with the technique of "combined variables" [13]. In the long-wave limit ($p \to 0$), those variables reduce to the hydrodynamic variables of macroscopic quantum hydrodynamics [14], while in the short-wave domain they correspond to the bosonic quasiparticle creation and annihilation operators.

On the other hand, according to numerous precise experimental data on neutron inelastic scattering [15]–[18] and to results in quantum evaporation of ^{4}He atoms [19], the maximal density ρ_0 of the single-particle Bose-Einstein condensate (BEC) in the ^{4}He Bose liquid even at very low temperatures $T \ll T_\lambda$ does not exceed 10% of the total density ρ of liquid ^{4}He, whereas the density of the SF component $\rho_s \to \rho$ at $T \to 0$ [20]. Such a low density of the BEC is implied by strong interaction between ^{4}He atoms and is an indication of the fact that the quantum structure of the part of the SF condensate in He II carrying the "excess" density $(\rho_s - \rho_0) \gg \rho_0$ calls for a more thorough investigation. The questions discussed in this paper are both those of the quantum structure of the SF state in a Bose liquid at $T = 0$ and the self-consistent calculation of the spectrum $E(p)$ of elementary excitations in the framework of renormalized field perturbation theory [9]–[13].

Our approach is based on the microscopic model developed in Refs. [21]–[22], of superfluidity of a Bose liquid with a suppressed BEC and an intensive pair coherent condensate (PCC), which can arise from a sufficiently strong effective attraction between bosons in some domains of momentum space (see below) and is analogous to the Cooper condensate in a Fermi liquid with attraction between fermions near the Fermi surface [23]. As a small parameter, one uses the ratio of the BEC density to the total Bose liquid density $(n_0/n) \ll 1$, unlike in the Bogolyubov theory [24] for a quasi-ideal Bose gas, in which the small parameter is the ratio of the number of supracondensate excitations to the density of the intensive BEC, $(n - n_0)/n_0 \ll 1$. Because of this, the SF state within the model at hand can be described by a "short" self-consistent system of Dyson-Belyaev equations for the normal and anomalous Green functions $\tilde{G}_{ik}$ and self-energy parts $\tilde{\Sigma}_{ij}(\mathbf{p}, \omega)$ without account for the diagrams of second and higher orders in the BEC density. In this case, the SF component ρ_s is a superposition of the "weak" single-particle BEC and an intensive "Cooperlike" PCC with coinciding phases (signs) of the corresponding order parameters. The pair interaction between bosons was chosen in the form of a finite repulsive potential in the "semitransparent", or "soft" spheres model, whose Fourier component $V(p)$ is an

oscillating sign-changing function of momentum transfer p due to mutual quantum diffraction of particles.

As a result of renormalization ("screening") of the initial interaction $V(p)$ due to multiparticle collective correlations, which are described by the boson polarization operator $\Pi(\mathbf{p},\omega)$, the interaction gets suppressed in the domains of momentum space where $V(p) > 0$, and enhanced where $V(p) < 0$. Such a suppression of repulsion and enhancement of attraction is implied by the negative sign of the real part of $\Pi(\mathbf{p},\omega)$ on the "mass shell" $\omega = E(p)$ for a decayless quasiparticle spectrum. It is shown that the integral contribution of the domains of effective attraction in the renormalized sign-changing interaction

$$\tilde{V}(\mathbf{p}) = V(p)\left[1 - V(p)\Re\,\Pi(\mathbf{p}, E(p))\right]^{-1}$$

can be sufficient for the formation of an intensive bosonic PCC in momentum space (although not for the formation of bound boson pairs in real space).

Self-consistent numerical calculations of the boson self-energy, polarization operator, pair order parameter, and quasiparticle spectrum at $T = 0$, involving an iteration scheme with the single fitting parameter—the value of the repulsion potential at $r = 0$, have allowed us to find conditions for the theoretical spectrum $E(p)$ to coincide with the experimentally observed elementary excitation spectrum in ^{4}He [25]–[30]. The roton minimum in the quasiparticle spectrum $E(p)$, which corresponds to a maximum in the structural form factor $S(q)$ of a Bose liquid, turns out to be directly associated with the first negative minimum of the Fourier component of the renormalized potential $\tilde{V}(\mathbf{p})$ of pair interaction between bosons.

2 Equations for the Green functions and self-energy parts in a Bose liquid with a suppressed BEC and an intensive PCC in the renormalized perturbation theory

The main difficulty of the microscopic description of the SF state of a Bose liquid with a nonzero BEC is the fact that applying perturbation theory directly [6] leads, as was shown in Refs. [9]–[12], to divergences and non-analyticities at small energies $\epsilon \to 0$ and momenta $\mathbf{p} \to 0$ and, as a consequence, to erroneous results in the calculations of different physical quantities. Thus, for example, for a Bose system with weak interaction, when the ratio of the mean potential energy $V(p_0)p_0^3$ (p_0 being a typical momentum transfer) to the corresponding kinetic energy $p_0^2/2m$ of the bosons is small, the zeroth-approximation polarization operator $\Pi(\mathbf{p},\omega)$ and the density-density response function $\tilde{\Pi}(\mathbf{p},\omega)$ calculated to the first order in the small parameter of interaction $\xi = mp_0V(p_0) \ll 1$, are logarithmically divergent at $p \to 0$, $\omega \to 0$, whereas the exact values $\Pi(0,0)$ and $\tilde{\Pi}(0,0)$ are finite [10]:

$$\Pi(0,0) = -\frac{\partial n}{\partial \mu} = -\frac{n}{mc^2}\ ; \qquad \tilde{\Pi}(0,0) = \frac{n}{m(c_B^2 - c^2)}\ , \tag{1}$$

where n is the total concentration of bosons, μ the chemical potential, $c_B = \sqrt{nV_0/m}$ the velocity of sound in the Bogolyubov approximation for a weakly nonideal Bose

gas [24], $V_0 \equiv V(0)$ the zero Fourier component of the potential, and c the speed of sound in the $\mathbf{p} \to 0$ limit for the spectrum of elementary excitations $\epsilon(p) \simeq c|\mathbf{p}|$ in the Belyaev theory [6]:

$$c = \sqrt{\Sigma_{12}(0,0)/m^*} \ . \qquad (2)$$

Here m^* is the effective mass of quasiparticles, which is determined by the relation [7]

$$\frac{1}{m^*} = \frac{2}{B}\left[\frac{1}{2m} + \frac{\partial\Sigma_{11}(0,0)}{\partial|\mathbf{k}|^2} - \frac{\partial\Sigma_{12}(0,0)}{\partial|\mathbf{k}|^2}\right] , \qquad (3)$$

where $\Sigma_{11}(0,0)$ is the normal self-energy part (at $k \to 0$, $\epsilon \to 0$), and

$$B = \left[1 - \frac{\partial\Sigma_{11}(0,0)}{\partial\epsilon}\right]^2 - \Sigma_{11}(0,0)\frac{\partial^2\Sigma_{12}(0,0)}{\partial\epsilon^2} + \frac{1}{2}\frac{\partial^2}{\partial\epsilon^2}\left[\Sigma_{12}(0,0)\right]^2 \ . \qquad (4)$$

The model, considered in Ref. [6], of a dilute Bose system of hard spheres with a small parameter $\beta = \sqrt{n/k_0^3} \ll 1$, in which there is a possibility to exclude the infinite repulsion by means of a summation of the "ladder" diagrams, leads to a finite value of $\Sigma_{12}(0,0)$ in the zeroth approximation in β:

$$\Sigma_{12}(0,0) = \frac{4\pi a_0}{m} n_0 \ , \qquad (5)$$

a_0 being the vacuum scattering amplitude and n_0 the concentration of bosons in the BEC ($\rho_0 = mn_0$).

At the same time, in Ref. [31], taking into account an exact thermodynamic equation

$$\frac{\partial\Sigma_{11}(0,0)}{\partial\epsilon} = -\left(\frac{\partial n_1}{\partial n_0}\right)_\mu = 1 - \frac{1}{n_0}\Sigma_{12}(0,0)\frac{dn_0}{d\mu} \ , \qquad (6)$$

where $n_1 = n - n_0$ is the concentration of supracondensate bosons, exact asymptotic relations were obtained for the normal and anomalous single-particle Green functions:

$$G_{11}(\mathbf{p} \to 0) = -G_{12}(\mathbf{p} \to 0) = \frac{n_0 mc^2}{n(\epsilon^2 - c^2\mathbf{p}^2 + i\delta)} \ ; \quad c^2 = \frac{n}{m}\frac{d\mu}{dn} \ . \qquad (7)$$

However, it was shown in Refs. [9]–[11] that at $\mathbf{p} = 0$, $\epsilon = 0$ the anomalous self-energy part is precisely equal to zero, $\Sigma_{12}(0,0) \equiv 0$. Problems then emerge with the determination of the velocity of sound (2) and the asymptotic formulas for $G_{11}(\mathbf{p},\epsilon)$ and $G_{12}(\mathbf{p},\epsilon)$ at $(\mathbf{p},\epsilon) \to 0$ [7]:

$$G_{11}(\mathbf{p} \to 0) = -G_{12}(\mathbf{p} \to 0) = \frac{\Sigma_{12}(0,0)}{B(\epsilon^2 - c^2\mathbf{p}^2 + i\delta)} \ , \qquad (8)$$

because at $\Sigma_{12}(0,0) = 0$, relations (4) and (6) reduce to identities

$$\frac{\partial\Sigma_{11}(0,0)}{\partial\epsilon} \equiv 1 \ , \qquad B \equiv 0 \ , \qquad (9)$$

so that Eqs. (2) and (8) with account for (3) contain uncertainties of the 0/0 type.

With the purpose of fixing these controversies, as well as the infrared divergences of $\Pi(\mathbf{p},\epsilon)$ and nonanalyticities in $\Sigma_{ij}(\mathbf{p},\epsilon)$ at $(\mathbf{p},\epsilon)\to 0$ emerging in the nonrenormalized theory, a renormalization procedure for the field perturbation theory was worked out in Ref. [12], employing the method of "combined variables" [13]. The perturbation theory built on such "adequate" field variables does not suffer from infrared divergences at $(\mathbf{p},\epsilon)\to 0$, whose source at $T=0$ is the divergence of long-wave quantum fluctuations (acoustic Goldstone oscillations). Such oscillations are associated with a spontaneous breakdown of continuous gauge and translational symmetries in the SF state of a Bose system with a uniform coherent condensate and corresponded to the hydrodynamic first sound in liquid ^{4}He, propagating with the velocity of $c_1\simeq 236$ m/s. The choice of combined variables [13] leads to the renormalized anomalous self-energy part $\tilde{\Sigma}_{12}(\mathbf{p},\epsilon)$ which does not vanish at $(\mathbf{p},\epsilon)=0$. Then one can formally restore all the results of the nonrenormalized field theory [6]–[7], but now in terms of the renormalized quantities $\tilde{G}_{ik}(\mathbf{p},\epsilon)$ and $\tilde{\Sigma}_{ik}(\mathbf{p},\epsilon)$, which do not contain singularities at $(\mathbf{p},\epsilon)\to 0$ (save for the pole part $\tilde{G}_{ik}(\mathbf{p},\epsilon)\sim|p|^{-2}$). In particular, the squared velocity of first sound c_1 at $T\to 0$ must be equal to

$$c_1^2=\frac{\tilde{\Sigma}_{12}(0,0)}{\tilde{m}^*}\ , \tag{10}$$

where the renormalized effective mass $\tilde{m}^*$ is determined by relations (3) and (4) with $\tilde{\Sigma}_{ik}(0,0)$ substituted for $\Sigma_{ik}(0,0)$ in Eq. (2). In view of the aforesaid, we will work with the combined variables [13],

$$\tilde{\Psi}(x)=\tilde{\Psi}_{\mathrm{L}}(x)+\tilde{\Psi}_{\mathrm{sh}}(x)\ , \tag{11}$$

where

$$\begin{gathered}\tilde{\Psi}_{\mathrm{L}}(x)=\sqrt{\langle\tilde{n}_{\mathrm{L}}\rangle}\left[1+\frac{\tilde{n}_{\mathrm{L}}-\langle\tilde{n}_{\mathrm{L}}\rangle}{2\,\langle\tilde{n}_{\mathrm{L}}\rangle}+i\tilde{\phi}_{\mathrm{L}}\right];\quad \tilde{\Psi}_{\mathrm{sh}}=\psi_{\mathrm{sh}}e^{-i\tilde{\phi}_{\mathrm{L}}}\ ;\\ \psi_{\mathrm{sh}}=\psi-\psi_{\mathrm{L}};\quad \psi_{\mathrm{L}}(\mathbf{r})=\frac{1}{\sqrt{V}}\sum_{|\mathbf{k}|<k_0}a_{\mathbf{k}}e^{i\mathbf{kr}}=\sqrt{\langle\tilde{n}_{\mathrm{L}}\rangle}e^{i\tilde{\phi}_{\mathrm{L}}}\ .\end{gathered} \tag{12}$$

Such an approach means that the separation of the Bose system into a macroscopic coherent condensate and a gas of supracondensate excitations is made not on the statistical level, like in the case of a weakly nonideal Bose gas [24], but on the level of ab initio field operators, which are used to construct a microscopic theory of the Bose liquid.

The system of Dyson-Belyaev equations [6]–[7], which allows one to express the normal $\tilde{G}_{11}$ and anomalous $\tilde{G}_{12}$ renormalized single-particle boson Green functions in terms of the respective self-energy parts $\tilde{\Sigma}_{11}$ and $\tilde{\Sigma}_{12}$, has the form (Fig. 1):

$$\tilde{G}_{11}(\mathbf{p},\epsilon)=\left[G_0^{-1}(-\mathbf{p},-\epsilon)-\tilde{\Sigma}_{11}(-\mathbf{p},-\epsilon)\right]/Z(\mathbf{p},\epsilon)\ ; \tag{13}$$

$$\tilde{G}_{12}(\mathbf{p},\epsilon)=\tilde{\Sigma}_{12}(\mathbf{p},\epsilon)/Z(\mathbf{p},\epsilon)\ . \tag{14}$$

Here

$$Z(\mathbf{p},\epsilon)=\left[G_0^{-1}(-\mathbf{p},-\epsilon)-\tilde{\Sigma}_{11}(-\mathbf{p},-\epsilon)\right]\left[G_0^{-1}(\mathbf{p},\epsilon)-\tilde{\Sigma}_{11}(\mathbf{p},\epsilon)\right]-|\tilde{\Sigma}_{12}(\mathbf{p},\epsilon)|^2\ ; \tag{15}$$

$$G_0^{-1}(\mathbf{p},\epsilon) = \left[\epsilon - \frac{\mathbf{p}^2}{2m} + \mu + i\delta\right] \qquad (\delta \to +0) \ , \tag{16}$$

where μ is the chemical potential of the quasiparticles, which satisfies the Hugengoltz-Pines relation [32]:

$$\mu = \tilde{\Sigma}_{11}(0,0) - \tilde{\Sigma}_{12}(0,0) \ . \tag{17}$$

Due to a strong hybridization of the single-particle and collective branches of elementary excitations in the Bose liquid with a finite BEC ($n_0 \neq 0$), the poles of the two-particle and all multiparticle Green functions coincide with the poles of the single-particle Green functions $\tilde{G}_{ik}(\mathbf{p},\epsilon)$ [7]–[8]. Therefore the spectrum of all elementary excitations with zero spirality is determined by the zeros of the function $Z(\mathbf{p},\epsilon)$:

$$E(p) = \left\{\left[\frac{\mathbf{p}^2}{2m} + \tilde{\Sigma}^s_{11}(\mathbf{p},E(p)) - \mu\right]^2 - |\tilde{\Sigma}_{12}(\mathbf{p},E(p))|^2\right\}^{1/2} + \tilde{\Sigma}^a_{11}(\mathbf{p},E(p)) \ , \tag{18}$$

where

$$\tilde{\Sigma}^{s,a}_{11}(\mathbf{p},\epsilon) = \frac{1}{2}\left[\tilde{\Sigma}_{11}(\mathbf{p},\epsilon) \pm \tilde{\Sigma}_{11}(-\mathbf{p},-\epsilon)\right] \ .$$

The $(+)$ and $(-)$ signs correspond to the symmetric $\tilde{\Sigma}^s_{11}$ and antisymmetric $\tilde{\Sigma}^a_{11}$ parts of $\tilde{\Sigma}_{11}$, respectively. Relation (17) ensures the acoustic dispersion law for the quasiparticles spectrum (18) at $\mathbf{p} \to 0$ with the sound velocity (10). As was shown in Ref. [21], for a Bose liquid with strong enough interaction between particles, when the BEC is strongly suppressed, one can, when defining $\tilde{\Sigma}_{ik}(\mathbf{p},\epsilon)$ in the form of a sequence of irreducible diagrams containing condensate lines, restrict oneself, with good precision, to the first (lowest) terms in the expansion over the small BEC density ($n_0 \ll n$). Such an approximation is exactly opposite to the Bogolyubov approximation [24] for a weakly nonideal Bose gas with an intensive BEC, when $n_0 \simeq n$. As a result, up to terms of first order in the small parameter $n_0/n \ll 1$, for a Bose liquid one gets the "trimmed" system of equations for $\tilde{\Sigma}_{ik}$ [21], [22] (see Fig.2):

$$\tilde{\Sigma}_{11}(\mathbf{p},\epsilon) = n_0\Lambda(\mathbf{p},\epsilon)\tilde{V}(\mathbf{p},\epsilon) + n_1 V(0) + \tilde{\Psi}_{11}(\mathbf{p},\epsilon) \ ; \tag{19}$$

$$\tilde{\Sigma}_{12}(\mathbf{p},\epsilon) = n_0\Lambda(\mathbf{p},\epsilon)\tilde{V}(\mathbf{p},\epsilon) + \tilde{\Psi}_{12}(\mathbf{p},\epsilon) \ , \tag{20}$$

where

$$\tilde{\Psi}_{ij}(\mathbf{p},\epsilon) = i\int \frac{d^3\mathbf{k}}{(2\pi)^3}\int \frac{d\omega}{2\pi} G_{ij}(\mathbf{k})\tilde{V}(\mathbf{p}-\mathbf{k},\epsilon-\omega)\Gamma(\mathbf{p},\epsilon,\mathbf{k},\omega) \ , \tag{21}$$

$$\tilde{V}(\mathbf{p},\epsilon) = V(p)\left[1 - V(p)\Pi(\mathbf{p},\epsilon)\right]^{-1} \ . \tag{22}$$

Here $V(p)$ is the Fourier component of the input potential of pair interaction of bosons, $\tilde{V}(\mathbf{p},\epsilon)$ is the renormalized ("screened"), due to multiparticle collective effects, Fourier component of the retarded (nonlocal) interaction; $\Pi(\mathbf{p},\epsilon)$ is the boson polarization operator:

$$\begin{gathered}\Pi(\mathbf{p},\epsilon) = i\int \frac{d^3\mathbf{k}}{(2\pi)^3}\int \frac{d\omega}{2\pi}\,\Gamma(\mathbf{p},\epsilon,\mathbf{k},\omega) \\ \times\left\{G_{11}(\mathbf{k},\omega)G_{11}(\mathbf{k}+\mathbf{p},\epsilon+\omega) + G_{12}(\mathbf{k},\omega)G_{12}(\mathbf{k}+\mathbf{p},\epsilon+\omega)\right\} \ ;\end{gathered} \tag{23}$$

$\Gamma(\mathbf{p},\,\epsilon;\,\mathbf{k},\,\omega)$ is the vertex part, which describes multiparticle correlations; $\Lambda(\mathbf{p},\epsilon) = \Gamma(\mathbf{p},\epsilon,0,0) = \Gamma(0,0,\mathbf{p},\epsilon)$, and n_1 is the number of supracondensate particles ($n_1 \gg n_0$), which is determined from the condition of conservation of the total number of particles:

$$n = n_0 + n_1 = n_0 + i\int \frac{d^3\mathbf{k}}{(2\pi)^3}\int \frac{d\omega}{2\pi}\, G_{11}(\mathbf{k},\omega)\ . \tag{24}$$

In the sequel, as well as in Ref. [21], in the integral relations (21) and (23) we will only consider the residues at poles of single-particle Green functions $\tilde{G}_{ij}(\mathbf{p},\epsilon)$, neglecting the contributions of eventual poles of the functions $\Gamma(\mathbf{p},\epsilon,\mathbf{k},\omega)$ and $\tilde{V}(\mathbf{p},\epsilon)$, which do not coincide with those of $\tilde{G}_{ij}(\mathbf{p},\epsilon)$. As a result, taking into account relations (13)–(20), Eqs. (21) on the mass shell $\epsilon = E(p)$ assume the following form (at $T=0$):

$$\begin{aligned}\tilde{\Psi}_{11}(\mathbf{p},E(p)) = \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\Gamma(\mathbf{p},E(p);\mathbf{k},E(k))\\ \times \tilde{V}(\mathbf{p}-\mathbf{k},E(\mathbf{p})-E(k))\left[\frac{A(\mathbf{k},E(k))}{E(k)}-1\right]\ ;\end{aligned} \tag{25}$$

$$\begin{aligned}\tilde{\Psi}_{12}(\mathbf{p},E(p)) = -\frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\Gamma(\mathbf{p},E(p);\mathbf{k},E(k))\tilde{V}(\mathbf{p}-\mathbf{k},E(p)-E(\mathbf{k}))\\ \times\,\frac{n_0\Lambda(\mathbf{k},E(k))\tilde{V}(\mathbf{k},E(k)) + \tilde{\Psi}_{12}(\mathbf{k},E(k))}{E(k)}\ ,\end{aligned} \tag{26}$$

where

$$\begin{aligned}E(p) = \left\{A^2(\mathbf{p},E(p)) - \left[n_0\Lambda(\mathbf{p},E(p))\tilde{V}(\mathbf{p},E(p)) + \tilde{\Psi}_{12}(\mathbf{p},E(p))\right]^2\right\}^{1/2}\\ + \frac{1}{2}\left[\tilde{\Psi}_{11}(\mathbf{p},E(p)) - \tilde{\Psi}_{11}(-\mathbf{p},-E(p))\right];\end{aligned} \tag{27}$$

$$\begin{aligned}A(\mathbf{p},E(p)) = n_0\Lambda(\mathbf{p},E(p))\tilde{V}(\mathbf{p},E(p))\\ + \frac{1}{2}\left[\tilde{\Psi}_{11}(\mathbf{p},E(p)) + \tilde{\Psi}_{11}(-\mathbf{p},-E(p))\right] - \tilde{\Psi}_{11}(0,0) + \tilde{\Psi}_{12}(0,0) + \frac{\mathbf{p}^2}{2m}\ .\end{aligned} \tag{28}$$

The total quasiparticle concentration in the Bose liquid is determined by the relation

$$n = n_0 + \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\frac{A(\mathbf{k},E(k))}{E(k)} - 1\right]\ . \tag{29}$$

From Eqs. (27) and (28) it follows that the quasiparticle spectrum $E(p)$, because of the analyticity of the functions $\tilde{\Psi}_{ij}(\mathbf{p},\epsilon)$, is acoustic at $p\to 0$, and its structure at $p\neq 0$ depends essentially on the features of the renormalized pair interaction of bosons. The theoretical spectrum (27) must be close to the experimental spectrum of elementary excitations in the ^{4}He Bose liquid [25]–[30] if this model is to be applicable for the description of the SF state in ^{4}He (see below).

Note that when the BEC is totally absent ($n_0 = 0$), equation (26) becomes homogeneous and degenerate with respect to the phase of $\tilde{\Psi}_{12}(p)$. It is then akin to the Bethe-Goldstone integral equation for a pair of particles in momentum space

$$\Psi(\mathbf{p}) = -\int \frac{d^3\mathbf{k}}{(2\pi)^3} V(\mathbf{p}-\mathbf{k}) \frac{\Psi(\mathbf{k})}{2E(k)-\Omega} , \tag{30}$$

with zero binding energy, $\Omega = 0$, which has a nontrivial solution $\Psi(\mathbf{p}) \neq 0$ only when there is attraction $V(\mathbf{q}) < 0$ in a broad enough region of momentum transfer $\mathbf{q}$. By virtue of this analogy, the function $\tilde{\Psi}_{12}(p)$ at $n_0 = 0$ can be taken to be the PCC order parameter [21]–[22], which describes condensation of boson pairs in momentum space (identical to the Cooper condensate of fermion pairs [23]).

The degeneracy of equation (26) with respect to the phase of $\tilde{\Psi}_{12}(p)$ at $n_0 \to 0$ allows for the condition of stability of the phonon spectrum $c_1^2 = \tilde{\Psi}_{12}(0)/\tilde{m}^* > 0$ to be met by means of choosing the appropriate sign (phase) of the pair order parameter $\tilde{\Psi}_{12}(0) > 0$. However, the model of the SF state with PCC and no BEC [21]–[22] implies a few paradoxes, such as a finite energy gap in the single-particle spectrum at $p = 0$, exponential asymptotic behavior of the pair correlation function $\left\langle \hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}') \right\rangle$ at $|\mathbf{r}-\mathbf{r}'| \to \infty$, half-integer quantum of circulation of the SF velocity $\kappa = \hbar/2m$ etc.

Indeed, at $\mathbf{p} \to 0$ and $n_0 \neq 0$, Eq. (26), due to the isotropic momentum dependence of the spectrum $E(p)$ and the functions $\tilde{V}(p) \equiv \tilde{V}(\mathbf{p}, E(p))$, $\Lambda(p) \equiv \Lambda(\mathbf{p}, E(p))$ and $\tilde{\Psi}_{12}(p) \equiv \tilde{\Psi}_{12}(\mathbf{p}, E(p))$ reduces to the form

$$\tilde{\Psi}_{12}(0) = -\frac{1}{(2\pi)^2} \int_0^\infty \frac{k^2 dk}{E(k)} \left[n_0 \Lambda^2(k) \tilde{V}^2(k) + \Lambda(k) \tilde{V}(k) \tilde{\Psi}_{12}(k) \right] . \tag{31}$$

The first integral addend on the right-hand side of Eq. (31) being always negative, the value of $\tilde{\Psi}_{12}(0)$ can be negative as well. The condition $\tilde{\Psi}_{12}(0) < 0$ means that the phase of the PCC is opposite to phase of the BEC, because $n_0 > 0$. Moreover, in this case, in spite of the condition $\Lambda(0)\tilde{V}(0) > 0$ (which ensures that the system is globally stable against a spontaneous collapse), at sufficiently small densities of the BEC, in accordance with Eq. (20), the values

$$\tilde{\Sigma}_{12}(0,0) = n_0 \Lambda(0) \tilde{V}(0) - |\tilde{\Psi}_{12}(0)| \tag{32}$$

become negative if $|\Psi_{12}(0)| > n_0 \Lambda(0) V(0)$, which corresponds to an instability in the phonon spectrum ($c_1^2 < 0$). However, if the pair interaction between bosons in a broad enough region of the momentum space has the character of attraction, i.e., $\Lambda(k)\tilde{V}(k) < 0$ at $k \neq 0$, and if the magnitude of that attraction is large enough (see below), the second (positive) addend on the right-hand side of Eq. (31) can outweigh the first (negative) one if the BEC density is small enough ($n_0 \ll n$). Then $\tilde{\Psi}_{12}(0)$ will be positive, and the phase of the PCC will coincide with phase of the BEC, so that $\tilde{\Sigma}_{12} > 0$ and $c_1^2 > 0$.

Since at $T = 0$ the density of the SF component ρ_s, on the one hand, coincides with the total density $\rho = mn$ of the Bose liquid and, on the other hand, ρ_s is proportional to $\tilde{\Sigma}_{12}(0,0)$ which plays the role of the SF order parameter, with

account for (20) and (31), one gets the following relations:

$$\rho_s \equiv \rho_0 + \tilde{\rho}_s = \beta m \frac{\tilde{\Sigma}_{12}(0,0)}{\Lambda(0)\tilde{V}(0)} = \beta m \left[n_0(1-\gamma) + \Psi \right] \tag{33}$$

where

$$\gamma = \frac{1}{(2\pi)^2 \Lambda(0)\tilde{V}(0)} \int_0^\infty \frac{k^2 dk}{E(k)} \left[\Lambda(k)\tilde{V}(k) \right]^2 , \tag{34}$$

$$\Psi = -\frac{1}{(2\pi)^2 \Lambda(0)\tilde{V}(0)} \int_0^\infty \frac{k^2 dk}{E(k)} \Lambda(k)\tilde{V}(k)\tilde{\Psi}_{12}(k) \ , \tag{35}$$

and β is a certain dimensionless constant. Since the density of the single-particle BEC is equal to $\rho_0 = mn_0$, we obtain $\beta = (1-\gamma)^{-1}$. This means that the density of the "Cooperlike" PCC is

$$\tilde{\rho}_s = mn_1 = \frac{m\Psi}{(1-\gamma)} \ , \tag{36}$$

where the concentration $n_1 = n - n_0$ is then determined from relation (29), and for liquid ^{4}He at $T \to 0$, in accordance with the experimental data [15]–[19], it should be approximately 90% of the full concentration of ^{4}He atoms in liquid helium $n = 2.17 \cdot 10^{22}$ cm^{-3}. Thus, the SF component of Bose liquid in this model at $T = 0$ is an effective coherent condensate [12] which is a superposition of the weak one-particle BEC and intensive PCC.

3 Choice of the pair interaction potential in the Bose liquid

To describe interaction of helium atoms in real space, various semi-empirical potentials are conventionally used, which describe strong repulsion at small distances and weak van der Waals attraction at large distances. However, most of those potentials are characterized by a strong divergence at $r \to 0$, like, for instance, the Lennard-Jones potential

$$U_{\mathrm{LJ}}(r) = \epsilon \left[\left(\frac{\sigma}{r} \right)^{12} - \left(\frac{\sigma}{r} \right)^{6} \right], \qquad r > r_c \ . \tag{37}$$

Such potentials are not suitable for the description of pair interaction in momentum space, since the respective Fourier components

$$V(p) = \int d^3r \, U(r) \exp\left(i\mathbf{pr} \right) = \frac{4\pi}{p} \int_0^\infty rU(r) \sin\left(pr \right) dr \tag{38}$$

are infinite, diverging at the lower limit. Lately, in the calculations of interatomic interaction and possible bound states, i.e., He_2 molecules, one uses more up-to-date potentials, like the Aziz potential [3]–[5]:

$$\begin{aligned} U_{\mathrm{A}}(r) &= A \exp(-\alpha r - \beta r^2) - F(r, r_0)(c_6 r^{-6} + c_8 r^{-8} + c_{10} r^{-10}) \ , \\ F(r, r_0) &= \begin{cases} \exp\left[-(r_0/r - 1)^2 \right] \sum_{k=0}^{2} , & r < r_0 \\ 1 \ , & r \geq r_0 \end{cases} \end{aligned} \tag{39}$$

where $A = 1.8443101 \times 10^5$ K, $\alpha = 10.43329537$ Å^{-1}, $\beta = 2.27965105$ Å^{-2}, $c_6 = 1.36745214$ K $\times$ Å^6, $c_8 = 0.42123807$ K $\times$ Å^8, $c_{10} = 0.17473318$ K $\times$ Å^{10}. Such potentials remain finite at $r = 0$ due to the nonanalytic exponential r dependence, which suppresses any power divergence at $r \to 0$ (see Fig. 3). However, employing its Fourier component in solving the nonlinear integral equations (25)–(26) is technically difficult while not conclusive by itself: In an intrinsically many-body problem like the one at hand, collective effects are certain to play an essential part and to render the subtleties in the shape of the two-body potential largely irrelevant. In order to retain the crucial features of the system yet not be overwhelmed by technical complications, we will utilize a model potential of "soft spheres" describing (unlike the "hard spheres" model [33]) finite repulsion in a certain bound region, which accounts for effects of mutual quantum diffraction of bosons in the Bose liquid.

In this context, consider a model potential in the form of a Fermi-type function in real space (Fig. 4a)

$$U_{\mathrm{F}}(r) = U_0 \left\{ \exp\left(\frac{r^2 - a^2}{b^2}\right) + 1 \right\}^{-1}, \tag{40}$$

which at $b = 0$ degenerates into a "step" of finite height U_0 at $r < a$. In this latter case the Fourier component is expressed in terms of the first order spherical Bessel function (see Fig. 5):

$$V(p) = V_0 \frac{j_1(pa)}{pa}; \qquad j_1(x) = \frac{\sin(x) - x\cos(x)}{x^2} . \tag{41}$$

where $V_0 \equiv 3V(0) = 4\pi U_0 a^3$. The same oscillating Fourier component is characteristic of a smooth potential $V(r)$ in the form of a Lindhardt-type function [23], having an infinite negative derivative at the inflection point $r = a$ (see Fig. 4b):

$$U_L(r) = \frac{U_0}{2} \left[1 + \frac{(1 - r^2/a^2)}{2r/a} \ln \left| \frac{a + r}{a - r} \right| \right]. \tag{42}$$

Formally, this problem is an inverse to the one of periodic oscillations of spin density in real space of interacting spins in metal (so-called Ruderman-Kittel-Kasui-Yoshida oscillations [34]).

Note that the amplitude of oscillations of the Fourier component for the Fermi type potentials (40) at $b \neq 0$ is damping exponentially with the increase of the parameter b, due to the decreasing absolute value of the negative derivative at the inflection point (see Fig. 5, inset).

Such oscillations of the Fourier component of the pair potential $U(r)$ in momentum space arise even in the absence of attraction in real space and are a consequence of quantum diffractional effects of mutual scattering of the particles. This means that the existence of negative values $V(p) < 0$, i.e., of effective attraction in some regions of momentum transfer p, is not directly associated with van der Waals forces, which are explicitly taken into account in the sign-changing (with respect to r) Lennard-Jones or Aziz potentials (see Fig. 3).

If one substitutes the oscillating potential (41) into the Bogolyubov spectrum of a dilute quasi-ideal Bose gas [24]

$$E_B(p) = \left\{ \frac{p^2}{2m} \left[\frac{p^2}{2m} + 2nV(p) \right] \right\}^{1/2} , \tag{43}$$

then, by choosing two parameters, V_0 and a, independently, one can achieve a rather satisfactory coincidence of the spectrum $E_B(p)$ with the elementary excitation spectrum $E_{\rm exp}(p)$ in liquid ^{4}He derived from neutron scattering experiments [25]–[29] (Fig. 6, solid and dotted curves). However, the spectrum (43) with the potential (41) turns out to be unstable at large values of V_0, because $E_B^2(p) < 0$ in some range of p (Fig. 6, dashed curve). Moreover, the Bogolyubov model of the quasi-ideal Bose gas with an intensive BEC at $T \to 0$ ($n_0 \to n$) is not applicable to the description of the Bose liquid with a strongly suppresed BEC ($n_0 \ll n$).

On the other hand, multiparticle collective effects in the Bose liquid, according to Eq. (22), lead to an essential renormalization of the pair interaction, which determines the normal and anomalous self-energy parts, Eqs. (19) and (20). An important feature of the renormalized interaction (22) is that in the regions of phase volume $(\mathbf{p}, \omega)$ where the real part of $\Pi(\mathbf{p}, \omega)$ is negative, the repulsion (when $V(p) > 0$) gets suppressed while the attraction (when $V(p) < 0$) gets effectively enhanced. This fact was first noted in Ref. [35] and used in Ref. [36] where the integral equations (25) and (26) were solved with the seed potential of the hard spheres model (see (37)). Note that in Ref. [36], a possibility for bound pairs of helium atoms to form not only in momentum space, but also in real space was discussed, which allowed one to interpret the anomalously large effective mass m_3^* of the dope ^{3}He atoms in the SF Bose liquid ^{4}He [37]–[38] as the mass of a bound ^{3}He-^{4}He pair, equal to $M = m_3 + m_4 = 7m_3/3$.

In this paper, we take into account the explicit momentum dependence of the real part of polarization operator $\Pi(\mathbf{p}, E(p))$, which can be represented as (see Appendix):

$$\begin{aligned} \Re\,\Pi(\mathbf{p}, E(p)) = \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\Gamma(\mathbf{p}, \mathbf{k})}{E(k) - E(|\mathbf{k}-\mathbf{p}|) - E(p)} \\ \times \left\{ \frac{F_-(\mathbf{k}, \mathbf{p})}{E(k)[E(k) + E(|\mathbf{k}-\mathbf{p}|) - E(p)]} - \frac{F_+(\mathbf{k}, \mathbf{p})}{E(k)[E(k) + E(|\mathbf{k}-\mathbf{p}|) + E(p)]} \right\} \end{aligned} \tag{44}$$

where

$$\begin{aligned} F_-(\mathbf{k}, \mathbf{p}) = \left[E(k) + \frac{\mathbf{k}^2}{2m} - \mu + \tilde{\Sigma}_{11}(\mathbf{k}, E(k)) \right] \\ \times \left[E(k) - E(p) + \frac{(\mathbf{k}-\mathbf{p})^2}{2m} - \mu + \tilde{\Sigma}_{11}(\mathbf{k}-\mathbf{p}, E(k) - E(p)) \right] \\ + \tilde{\Sigma}_{12}(\mathbf{k}, E(k)) \tilde{\Sigma}_{12}(\mathbf{k}-\mathbf{p}, E(k) - E(p)) \, , \end{aligned} \tag{45}$$

$$
\begin{aligned}
F_+(\mathbf{k},\mathbf{p}) = &\left[E(|\mathbf{k}-\mathbf{p}|) + \frac{(\mathbf{k}-\mathbf{p})^2}{2m} - \mu + \tilde{\Sigma}_{11}(\mathbf{k}-\mathbf{p}, E(|\mathbf{k}-\mathbf{p}|))\right] \\
&\times \left[E(|\mathbf{k}-\mathbf{p}|) + E(p) + \frac{\mathbf{k}^2}{2m} - \mu + \tilde{\Sigma}_{11}(\mathbf{k}, E(|\mathbf{k}-\mathbf{p}|) + E(p))\right] \\
&+ \tilde{\Sigma}_{12}(\mathbf{k}-\mathbf{p}, E(|\mathbf{k}-\mathbf{p}|))\tilde{\Sigma}_{12}(\mathbf{k}, E(|\mathbf{k}-\mathbf{p}|) + E(p)) \ .
\end{aligned} \tag{46}
$$

As follows from Eq. (44), if the quasiparticle spectrum $E(p)$ is stable with respect to decays into a pair of quasiparticles [7],[39], i.e., if for all $\mathbf{p}$ and $\mathbf{k}$ the following conditions are fulfilled:

$$
E(p) < E(k) + E(|\mathbf{k}-\mathbf{p}|) \ ; \qquad E(k) < E(p) + E(|\mathbf{k}-\mathbf{p}|), \tag{47}
$$

the common denominator in front of the curly braces is always negative,

$$
E(k) - E(|\mathbf{k}-\mathbf{p}|) - E(p) < 0 \ , \tag{48}
$$

whereas the denominator of the first term in the curly braces is always positive,

$$
E(k) + E(|\mathbf{k}-\mathbf{p}|) - E(p) > 0 \tag{49}
$$

and smaller than the positive denominator of the second term

$$
E(k) + E(|\mathbf{k}-\mathbf{p}|) + E(p) > 0 \ . \tag{50}
$$

This means that the integrand of Eq. (44) is negative if the functions $F_\pm(\mathbf{k},\mathbf{p})$ are positive. According to the numerical calculations [36], [40] in the framework of the "hard" and "soft" sphere models, $F_\pm(\mathbf{k},\mathbf{p}) > 0$ for all $\mathbf{k}$ and $\mathbf{p}$, so that the real part of $\Pi(\mathbf{p}, E(p))$ is negative, because $\Gamma > 0$ (see below). One should note that the actual experimental spectrum of elementary excitations in liquid ^{4}He is decaying at small enough momenta. However, this will not change the negative sign of $\Pi(\mathbf{p})$ due to the integral character of expression (44) (cf. [40]). Note also that here we do not take into account the imaginary part of $\Pi(\mathbf{p},\omega)$, which determines the damping of quasiparticles and the dynamical structure factor (see Appendix).

4 The iterative scheme of calculation of the quasiparticle spectrum

In order to calculate the quasiparticle spectrum within the model of a Bose liquid with a suppressed BEC and intensive PCC being considered, at first, using Eqs. (25) and (26), a numerical calculation in the first approximation of the functions $\Phi_1(p) \equiv \tilde{\Psi}_{11}(\mathbf{p}, E_0(\mathbf{p}))$ and $\Psi_1(p) \equiv \tilde{\Psi}_{12}(\mathbf{p}, E_0(\mathbf{p}))$, was conducted. For the zeroth approximation, the Bogolyubov spectrum $E_0(p) = E_B(p)$ and the "screened" potential (22) with account for Eq. (41) were taken:

$$
\tilde{V}_0(p) = \frac{V_0 j_1(pa)}{pa - V_0 \Pi_0 j_1(pa)} \tag{51}
$$

at some constant negative value of Π_0. Then, using the functions $\Phi_1(p)$ and $\Psi_1(p)$ obtained, the first approximation for the polarization operator $\Pi_1(p)$ was calculated,

using Eqs. (44)–(46) at $\Gamma = 1$. Here, too, the Bogolyubov spectrum (43), which is the best fit to the empiric spectrum $E_{\text{exp}}(p)$ for liquid ^{4}He (see Fig. 6) was chosen as the zeroth approximation for $E(p)$. The limiting value $\Pi_1(0)$ was compared with the exact thermodynamic value of the polarization operator of the ^{4}He Bose liquid at $p = 0$ and $\omega = 0$ [cf. (1)], which determines the compressibility of the Bose system: $\Pi(0, 0) = -n/mc_1^2$.

The absolute value $|\Pi(0, 0)|$ turned out to be almost 1.5 times greater than the calculated value $|\Pi_1(0)|$. This provides an estimate of the vertex Γ at $p = 0$ in the first approximation as $\Gamma_1 \equiv \Lambda_1 \simeq 1.5$. The second approximation $\Phi_2(p)$ and $\Psi_2(p)$ was obtained from Eqs. (25), (26) with the constant value $\Gamma_1 \equiv \Lambda_1$ and the first approximation for the renormalized potential, Eq. (44):

$$\tilde{V}_1(p) = \frac{V_0 j_1(pa)}{pa - V_0 \Pi_1(p) j_1(pa)} \,. \tag{52}$$

Such an iterative procedure was repeated four to six times and used to improve precision in the calculation of the polarization operator. At each stage, equations (27) and (28) were used to reproduce the quasiparticle spectrum $E(p)$, and the rate of convergence of the iterations was watched, as well as the degree of proximity of $E(p)$ to the empirical spectrum $E_{\text{exp}}(p)$.

The only fitting parameter in these calculations was the amplitude V_0 of the seed potential (41) at the value of $a = 2.44$ Å, which is equal to twice the quantum radius of the ^{4}He atom. The BEC density, in accordance with the experimental data [15]–[19], was fixed at $n_0 = 9\% n = 1.95 \cdot 10^{21}\,\text{cm}^{-3}$. The computation has resulted in a quite satisfactory agreement of the theoretical spectrum $E(p)$ with $E_{\text{exp}}(p)$. Figure 8 depicts the momentum dependence of $\Pi(p)$ as obtained with five iterations, while Fig. 9 shows the self-consistent p dependences of $\Phi(p)$, $\Psi(p)$, and $A(p)$ obtained from Eqs. (25), (26), and (28). One notices that the functions $\Phi(p)$ and $\Pi(p)$ are negative at all p, whereas $\Psi(p)$ and $A(p)$ are positive. The locations of the deep minima of $\Phi(p)$ and $A(p)$ practically coincide with the location of the minimum of the potential (22) (see Fig. 7).

Finally, in Fig. 10, the solid curve is the theoretical quasiparticle spectrum $E(p)$ obtained from Eq. (27), and the dots are the experimental spectrum obtained from data on inelastic neutron scattering in ^{4}He [25]–[29]. In the calculation of $E(p)$, the fitting parameter V_0 was chosen in such a way that the phase velocity of quasiparticles $E(p)/p$ at $p \to 0$ coincide with the speed of the first hydrodynamic sound $c_1 \simeq 236$ m/s in liquid ^{4}He. This value of V_0 corresponds to the repulsion potential of the "soft" spheres model $U_0 = V_0/(4\pi a^3) = 1552$ K at $a = 2.44$ Å. We see that there is a satisfactory agreement of $E(p)$ with $E_{\text{exp}}(p)$ in the region $p \leq 2.5\ \text{Å}^{-1}$. For $p > 2.5\ \text{Å}^{-1}$, the theoretical spectrum $E(p)$ lies somewhat higher than $E_{\text{exp}}(p)$, which, apparently, has to do with the fact that the vertex function $\Gamma(\mathbf{k}, \mathbf{p})$, a decreasing function of p, was replaced with a constant value $\tilde{\Gamma} \simeq 1.5$ for all p.

Of course, the value of $E(p)$ at large momenta should not exceed the doubled value of the roton gap $\Delta_r = 8.61$ K lest the spectrum becomes decaying [39]. To check this, we have approximated the vertex $\Lambda(p) = \Gamma(0, p)$ with a slowly decreasing function, falling down from $\tilde{\Gamma} = 1.5$ to $\tilde{\Gamma} = 1.1$ on the interval $2.1\ \text{Å}^{-1} < p <$

$3.8\ \text{Å}^{-1}$. The resulting theoretical spectrum is shown in Fig. 11, together with the experimental data [25]–[29] (light circles) and the latest results [30] (asterisks) of measurements of the spectrum at $2\ \text{Å}^{-1} < p < 3.6\ \text{Å}^{-1}$, beyond the roton minimum. Evidently, such an approximation for the vertex part yields a much better agreement of the theoretical and experimental spectra at large values of momentum. Note also that the self-consistency of this model is corroborated by the fact that the theoretical value of total particle concentration calculated from Eq. (29), $n_{\text{th}} = 2.12\cdot 10^{22}\ \text{cm}^{-3}$, is quite close to the experimental value for liquid ^{4}He, $n = 2.17\cdot 10^{22}\ \text{cm}^{-3}$ (at $n_0 = 9\%\, n$). On the other hand, the concentration n_1 of supracondensate particles, calculated from Eqs. (34)–(36) at the values of the parameters indicated, is about $0.93\, n$, which is also in good accordance with experiment, taking into account that the BEC density is determined up to $\pm 0.01\, n$. Also, when formulating the approximate theoretical model, quadratic terms in the small parameter $n_0/n \ll 1$ in Eqs. (19)–(20) were omitted, which also introduces an error of the order of 1% [15]–[20].

5 Conclusions

Thus, the model of the SF state of a Bose liquid with a single-particle BEC suppressed because of interaction and an intensive PCC, based upon a renormalized field perturbation theory with combined variables [11]–[13], allows one to obtain a self-consistent "trimmed" system of nonlinear integral equations for the self-energy parts $\tilde{\Sigma}_{ij}(p, \epsilon)$, by means of truncating the infinite series in the small density of the BEC ($n_0/n \ll 1$). By the same token, one can work out a self-consistent microscopic theory of a superfluid Bose liquid and perform an ab initio calculation of the spectrum of elementary excitations $E(p)$, starting from realistic models of pair interaction potential $U(r)$ possessing finite Fourier components. It is shown that for a repulsive potential in the framework of the "soft spheres" model, the Fourier component $V(p)$ is an oscillating sign-changing function of momentum transfer p. This means that in certain regions of momentum space at $p \neq 0$ there is an effective attraction between bosons, $V(p) < 0$, which has nothing to do with van der Waals forces and has a quantum mechanical diffraction nature. That attraction gets substantially enhanced due to multiparticle collective effects of renormalization ("screening") of the initial interaction, which are described by the boson polarization operator $\Pi(\mathbf{p}, \omega)$. The enhancement of the attraction happens because on the "mass shell" $\omega = E(p)$, the real part of $\Pi(\mathbf{p}, E(p))$ is negative in the whole region of momentum where the quasiparticle spectrum $E(p)$ is stable with respect to decay [39]. It is necessary to emphasize that this negative sign of $\Re\,\Pi(\mathbf{p}, E(p))$ is only characteristic of Bose systems, in which the single-particle and collective spectra coincide with each other and are measured from the common zero of energy, unlike the Fermi systems, in which the single-particle excitation spectrum begins at the Fermi energy, due to the Pauli principle. Therefore, in the ^{3}He Fermi liquid there can be no corresponding effective enhancement of the negative values of the same "input" interaction potential $V(p)$, so that the formation of Cooper pairs is only possible for nonzero orbital momenta, due to the true weak van der Waals attrac-

tion between fermions [41]–[42]. Apparently, it is this fact that has to do with the critical temperatures of the SF transition in ^{4}He and ^{3}He differing by three orders of magnitude.

The rather strong pair attraction of bosons in momentum space for $\Re\,\Pi(p,\omega) < 0$ forms an intensive PCC, which, together with a weak BEC, constitutes a single coherent condensate, making up the microscopic foundation of the SF component of the Bose liquid $\rho_s \sim \tilde{\Sigma}_{12}(0,0)$. On the other hand, the oscillating nature of the renormalized Fourier component of the potential $\tilde{V}(p)$ (see Fig. 5 and Fig. 7) leads to a nonmonotonic behavior of momentum dependences of the mass operators $\tilde{\Sigma}_{11}(p, E(p))$ and $\tilde{\Sigma}_{12}(p, E(p))$, and, as a consequence, to the emergence of a roton minimum in the quasiparticle spectrum $E(p)$, which is directly connected with the deepest first negative minimum of $\tilde{V}(p)$.

It is necessary to emphasize that the amplitude of the "soft spheres" model repulsion potential obtained, $U_0 = 1552$ K, which corresponds to very good agreement between the theoretical quasiparticle spectrum $E(p)$ and the experimental spectrum of elementary excitations in ^{4} He is smaller than the value of the Aziz type potential at $r \to 0$ (see Fig. 3, inset). It is a result of strong quantum diffraction effects in the Bose liquid, because the average distance between particles is of the order of or less than the de Broglie wavelength of the bosons.

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7 Appendix

The polarization operator (19) can be calculated without account for the vertex part Γ, making use of expressions (3)–(6) in the form

$$\Pi(\mathbf{p},\omega) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[L_{11}(\mathbf{p},\mathbf{k},\omega) + L_{12}(\mathbf{p},\mathbf{k},\omega)\right] \ , \tag{A.1}$$

where

$$L_{ij}(\mathbf{p},\mathbf{k},\omega) = i\oint \frac{dz}{2\pi}\tilde{G}_{ij}(\mathbf{k},z)\tilde{G}_{ij}(\mathbf{k}-\mathbf{p},z-\omega) \ . \tag{A.2}$$

Assume that the Green functions $\tilde{G}_{ij}$ have only one pole within the integration contour and are equal to

$$\tilde{G}_{11}(\mathbf{k},\epsilon) = \frac{\epsilon + \frac{k^2}{2m} - \mu + \tilde{\Sigma}_{11}(-\mathbf{k},-\epsilon)}{\epsilon^2 - E^2(k) + i\delta} \ ; \tag{A.3}$$

$$\tilde{G}_{12}(\mathbf{k},\epsilon) = \frac{\tilde{\Sigma}_{12}(\mathbf{k},\epsilon)}{\epsilon^2 - E^2(k) + i\delta} \qquad (\delta \to 0) \ . \tag{A.4}$$

Calculating the integrals (A.2) with account for the poles at the points $\epsilon = E(k)$ and $\epsilon = E(|\mathbf{k}-\mathbf{p}|)+\omega$ yields

$$L_{11}(\mathbf{p},\mathbf{k},\omega) = \frac{1}{2\left[E(k)-E(|\mathbf{k}-\mathbf{p}|)-\omega\right]}\left\{\left[E(k)+\frac{\mathbf{k}^2}{2m}-\mu+\tilde{\Sigma}_{11}(\mathbf{k},E(k))\right]\right.$$

$$\times\frac{\left[E(k)-\omega+\frac{(\mathbf{k}-\mathbf{p})^2}{2m}-\mu+\tilde{\Sigma}_{11}(\mathbf{k}-\mathbf{p},E(k)-\omega)\right]}{E(k)\left[E(k)+E(|\mathbf{k}-\mathbf{p}|)-\omega\right]}$$

$$-\left[E(|\mathbf{k}-\mathbf{p}|)+\frac{(\mathbf{k}-\mathbf{p})^2}{2m}-\mu+\tilde{\Sigma}_{11}(\mathbf{k}-\mathbf{p},E(|\mathbf{k}-\mathbf{p}|))\right]$$

$$\left.\times\frac{\left[E(|\mathbf{k}-\mathbf{p}|)+\omega+\frac{\mathbf{k}^2}{2m}-\mu+\tilde{\Sigma}_{11}(\mathbf{k},E(|\mathbf{k}-\mathbf{p}|)+\omega)\right]}{E(|\mathbf{k}-\mathbf{p}|)\left[E(k)+E(|\mathbf{k}-\mathbf{p}|)+\omega\right]}\right\}, \tag{A.5}$$

$$L_{12}(\mathbf{p},\mathbf{k},\omega) = \frac{1}{2\left[E(k)-E(|\mathbf{k}-\mathbf{p}|)-\omega\right]}\left\{\frac{\tilde{\Sigma}_{12}(\mathbf{k},E(k))\tilde{\Sigma}_{12}(\mathbf{k}-\mathbf{p},E(k)-\omega)}{E(k)\left[E(k)+E(|\mathbf{k}-\mathbf{p}|)-\omega\right]}\right.$$

$$\left.-\frac{\tilde{\Sigma}_{12}(\mathbf{k},E(|\mathbf{k}-\mathbf{p}|)+\omega)\tilde{\Sigma}_{12}(\mathbf{k}-\mathbf{p},E(|\mathbf{k}-\mathbf{p}|))}{E(|\mathbf{k}-\mathbf{p}|)\left[E(k)+E(|\mathbf{k}-\mathbf{p}|)+\omega\right]}\right\} \tag{A.6}$$

In the statistical limit ($\omega\to 0, \mathbf{p}\to 0$), expression (A.5) reduces to

$$L_{11}(0,\mathbf{k},0) = -\frac{1}{4}\left\{\frac{1}{E^2(k)}\left[\epsilon(k)+\frac{\mathbf{k}^2}{2m}-\mu+\tilde{\Sigma}_{11}(\mathbf{k},\epsilon(k))\right]^2\right.$$
$$\left.+\left[\frac{2}{\epsilon(k)}\left(1+\frac{\partial\tilde{\Sigma}_{11}(\mathbf{k})}{\partial\epsilon}\right)-\frac{k}{m\epsilon(k)}\frac{1}{\frac{\partial\epsilon(k)}{\partial k}}\right]\left[\epsilon(k)+\frac{\mathbf{k}^2}{2m}-\mu+\tilde{\Sigma}_{11}(\mathbf{k},\epsilon(k))\right]\right\} \tag{A.7}$$

It follows that in a large region of momentum space, $I_{11}(0,\mathbf{k},0)<0$. The same result is obtained for the function (A.6) at $p=0$ and $\omega=0$, i.e., $L_{12}(0,\mathbf{k},0)<0$, so that the static bosonic polarization operator $\Pi(0,0)$ is negative, which corresponds to a suppression of the "screened" repulsion at $\mathbf{p}\to 0$. From Eqs. (A.5) and (A.6) it can also be seen that on the "mass shell" $\omega = E(p)$, the integrals L_{11} and L_{12} remain negative in a wide region of momentum space because of the negative sign of the common denominator $E(k)-E(|\mathbf{k}-\mathbf{p}|)-E(p)<0$ and the positive sign of the denominator $E(k)+E(|\mathbf{k}-\mathbf{p}|)-E(p)>0$ due to the fact that the quasiparticle spectrum $E(p)$ is decayless [conditions (47)].

Thus, the real part of the polarization operator (A.1) at $\omega = E(p)$ is negative on the whole range of p. In order to determine the imaginary part of $\Pi(\mathbf{p},\omega)$, which describes Landau quantum damping of bosons, one has to calculate the main value of the following integral:

$$L(\mathbf{p},\mathbf{k},\omega) = \frac{i}{2\pi}\,\mathrm{V.p.}\int_{-\infty}^{\infty}\frac{d\epsilon}{\left[\epsilon^2-E^2(k)\right]\left[(\epsilon-\omega)^2-E^2(\mathbf{k}-\mathbf{p})\right]}$$
$$\left\{\left[\epsilon+\sigma(\mathbf{k},\epsilon)\right]\left[\epsilon-\omega+\sigma(\mathbf{k}-\mathbf{p},\epsilon-\omega)\right]+\tilde{\Sigma}_{12}(\mathbf{k},\epsilon)\tilde{\Sigma}_{12}(\mathbf{k}-\mathbf{p},\epsilon-\omega)\right\}, \tag{A.8}$$

where

$$\sigma(\mathbf{k},\epsilon) = \frac{\mathbf{k}^2}{2m} - \mu + \tilde{\Sigma}_{11}(\mathbf{k},\epsilon)$$

Factoring the denominators in the integrand of(A.8) into simple fractions, one obtains

$$L(\mathbf{p},\mathbf{k},\omega) = \frac{i}{2\pi E(|\mathbf{k}-\mathbf{p}|)}\ \text{V.p.} \int\limits_{-\infty}^{\infty} d\epsilon \left[M_+(\mathbf{p},\mathbf{k},\omega,\epsilon) - M_-(\mathbf{p},\mathbf{k},\omega,\epsilon)\right] \qquad \text{(A.9)}$$

where

$$M_\pm(\mathbf{p},\mathbf{k},\omega,\epsilon) = \frac{\epsilon C_\pm(\mathbf{p},\mathbf{k},\omega,\epsilon) + D_\pm(\mathbf{p},\mathbf{k},\omega,\epsilon)}{\epsilon^2 - E^2(k)} + \frac{C_\pm(\mathbf{p},\mathbf{k},\omega,\epsilon)}{\epsilon - \omega \mp E(|\mathbf{k}-\mathbf{p}|)}\ ; \qquad \text{(A.10)}$$

$$C_\pm(\mathbf{p},\mathbf{k},\omega,\epsilon) = \frac{1}{E^2(k) - [\omega \pm E(\mathbf{k}-\mathbf{k})]^2}\left\{[E(k)+\sigma(\mathbf{k},\epsilon)]\,\sigma(\mathbf{k}-\mathbf{p},\epsilon-\omega)\right.$$
$$\left. + [E(|\mathbf{k}-\mathbf{p}|)+\omega]\left[\sigma(\mathbf{k}-\mathbf{p},\epsilon-\omega)-\omega\right] + \tilde{\Sigma}_{12}(\mathbf{k},\epsilon)\tilde{\Sigma}_{12}(\mathbf{k}-\mathbf{p},\epsilon-\omega)\right\}\ ; \qquad \text{(A.11)}$$

$$D_\pm(\mathbf{p},\mathbf{k},\omega,\epsilon) = \sigma(\mathbf{k},\epsilon) + \sigma(\mathbf{k}-\mathbf{p},\epsilon-\omega) - \omega + [\omega \pm E(|\mathbf{k}-\mathbf{p}|)]\,C_\pm(\mathbf{p},\mathbf{k},\epsilon,\omega)\ . \qquad \text{(A.12)}$$

If one neglects the explicit ϵ dependence of $\tilde{\Sigma}_{ij}(\mathbf{k},\epsilon)$, the integral (A.9) vanishes. On the other hand, from Eq. (A.10) it follows that the nonvanishing contribution into $L(\mathbf{p},\mathbf{k},\omega)$ is given by the odd in ϵ parts of the functions $C_\pm(\mathbf{p},\mathbf{k},\omega,\epsilon)$ and the even in ϵ parts of the functions $D_\pm(\mathbf{p},\mathbf{k},\omega,\epsilon)$. Since $\tilde{\Sigma}_{12}(\mathbf{p},\mathbf{k})$ is an even function of ϵ, and $\tilde{\Sigma}_{11}(\mathbf{p},\mathbf{k})$ contains both an even $\tilde{\Sigma}^s_{11}(\mathbf{p},\mathbf{k})$ and an odd $\tilde{\Sigma}^a_{12}(\mathbf{p},\mathbf{k})$ part, the expressions for $C_\pm$ and $D_\pm$, which provide for nonzero values of $\Im\Pi(\mathbf{k},\omega)$, can be cast into the form

$$\tilde{C}_\pm(\mathbf{p},\mathbf{k},\omega,\epsilon) = \frac{1}{E^2(k) - [\omega \pm E(\mathbf{k}-\mathbf{k})]^2}\left\{\tilde{\Sigma}^a_{11}(\mathbf{k}-\mathbf{p},\epsilon-\omega)\right.$$
$$\times\left[E(k) + E(|\mathbf{k}-\mathbf{p}|) + \omega + \tilde{\Sigma}^s_{11}(\mathbf{k},\epsilon) - \mu + \frac{\mathbf{k}^2}{2m}\right] \qquad \text{(A.13)}$$
$$\left. + \tilde{\Sigma}^a_{11}(\mathbf{k},\epsilon)\left[\tilde{\Sigma}^s_{11}(\mathbf{k}-\mathbf{p},\epsilon-\omega) - \mu + \frac{(\mathbf{k}-\mathbf{p})^2}{2m}\right]\right\}\ ;$$

$$\tilde{D}_\pm(\mathbf{p},\mathbf{k},\omega,\epsilon) = \tilde{\Sigma}^s_{11}(\mathbf{k},\epsilon) + \tilde{\Sigma}^s_{11}(\mathbf{k}-\mathbf{p},\epsilon-\omega) - 2\mu + \frac{\mathbf{k}^2}{2m} + \frac{(\mathbf{k}-\mathbf{p})^2}{2m} - \omega$$
$$+ [\omega \pm E(|\mathbf{k}-\mathbf{p}|)]\left\{\tilde{\Sigma}^s_{11}(\mathbf{k}-\mathbf{p},\epsilon-\omega)\left[E(k) + E(|\mathbf{k}-\mathbf{p}|) + \omega + \tilde{\Sigma}^s_{11}(\mathbf{k},\epsilon) - \mu + \frac{\mathbf{k}^2}{2m}\right]\right.$$
$$\left. + \tilde{\Sigma}^a_{11}(\mathbf{k},\epsilon)\tilde{\Sigma}^a_{11}(\mathbf{k}-\mathbf{p},\epsilon-\omega) + \tilde{\Sigma}_{12}(\mathbf{k},\epsilon)\tilde{\Sigma}_{12}(\mathbf{k}-\mathbf{p},\epsilon-\omega)\right\}\ . \qquad \text{(A.14)}$$

At the same time, a nonzero value of $\Im\,\Pi(\mathbf{p},\omega)$ implies that the retarded renormalized potential (22) becomes complex:

$$\Re\,\tilde{V}(\mathbf{p},\omega) = V(p)\frac{1 - V(\mathbf{p})\Re\,\Pi(\mathbf{p},\omega)}{\left[1 - V(\mathbf{p})\Re\,\Pi(\mathbf{p},\omega)\right]^2 + \left[V(\mathbf{p})\Im\Pi(\mathbf{p},\omega)\right]^2}\ ; \tag{A.15}$$

$$\Im\,\tilde{V}(\mathbf{p},\omega) = \frac{-V^2(\mathbf{p})\Im\,\Pi(\mathbf{p},\omega)}{\left[1 - V(\mathbf{p})\Re\,\Pi(\mathbf{p},\omega)\right]^2 + \left[V(\mathbf{p})\Im\,\Pi(\mathbf{p},\omega)\right]^2}\ . \tag{A.16}$$

Therefore, the functions $\tilde{\Psi}_{11}(\mathbf{p},\omega)$ and $\tilde{\Psi}_{12}(\mathbf{p},\omega)$ become complex, too, and so does the quasiparticle spectrum $\tilde{E}(p) = E(p) + i\gamma(\mathbf{p})$, where γ is the decrement of damping.

The dynamical structure factor

$$S(\mathbf{p},\omega) = -\frac{1}{\pi n}\Im\,\tilde{\Pi}(\mathbf{p},\omega)\ ,$$

which is measured in neutron scattering experiments and is usually determined by the imaginary part of the density-density correlation function $\tilde{\Pi}(\mathbf{p},\omega)$ (see, for instance, [43]), can be also obtained as the imaginary part of the "screened" interaction

$$S(\mathbf{p},\omega) = -\frac{1}{\pi V(q)}\Im\,\tilde{V}(\mathbf{p},\omega)\ .$$

On the "mass shell", the form factor $S(p) \equiv S(p, E(p))$ has a maximum in the same region of p where the potential $\tilde{V}(\mathbf{p},\omega)$ and correspondingly the quasiparticle spectrum $E(p)$ have a minimum, in accordance with the general conection beetween $E(p)$ and $S(p)$ (see [44]–[45]).

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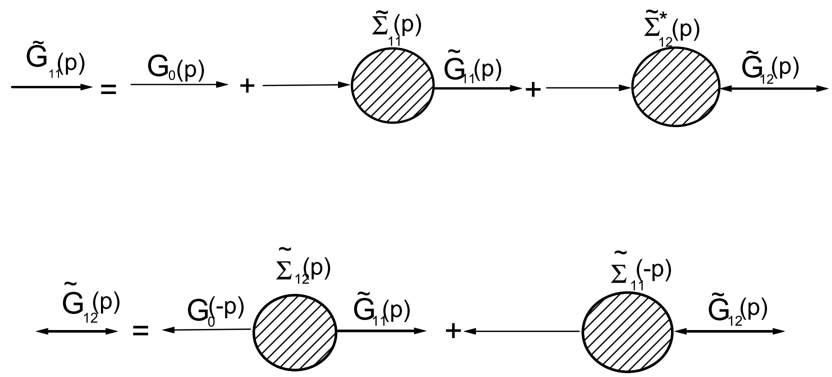


Fig. 1. The diagram representation of the Dyson–Belyaev equations for the normal $\tilde{G}_{11}$ and anomalous $\tilde{G}_{12}$ Green functions of bosons in the superfluid Bose system.

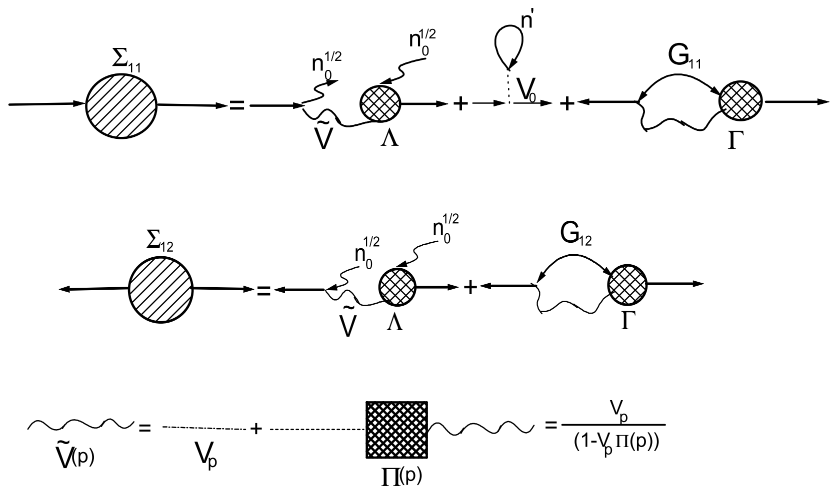


Fig. 2. The diagram representation of the nonlinear integral equations for the normal $\tilde{\Sigma}_{11}$ and anomalous $\tilde{\Sigma}_{12}$ self-energy parts in the Bose liquid with account for terms of first order in the small density of the single-particle BEC ($n_0 \ll n$) and of the equation for the renormalized ("screened") potential of pair interaction $\tilde{V}$.

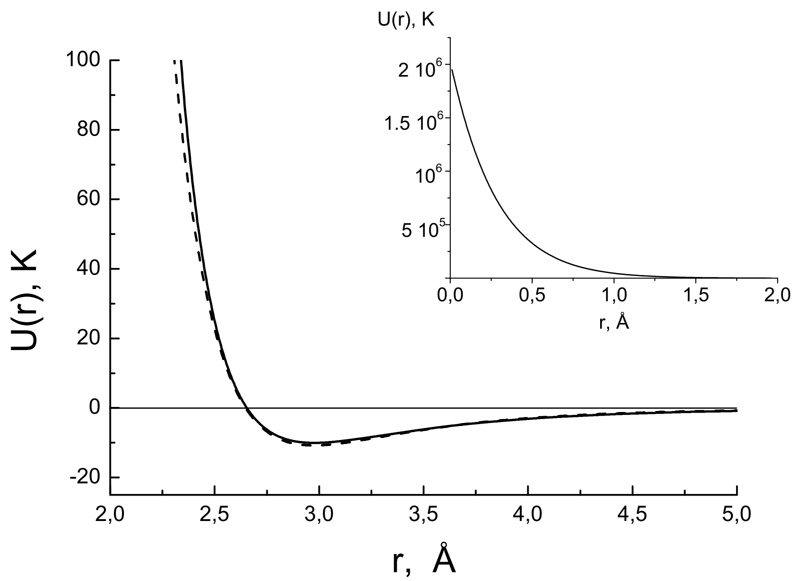


Fig. 3. The radial dependences of the regularized Aziz potential (39) (solid curve) and the Lennard-Jones potential (37) (dashed curve). Inset, the Aziz potential at small distances, down to $r = 0$.

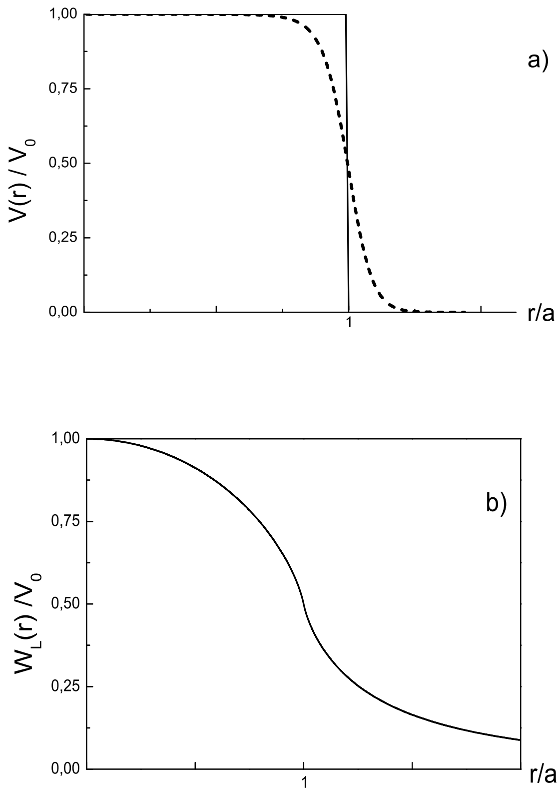


Fig. 4. The finite repulsion potential in the model of "soft spheres" in the form of (a) a Fermi "step" (40) (solid curve corresponds to $b = 0$, dashed curve—to $b = 0.5a$), and (b) a Lindhardt function (42) in real space.

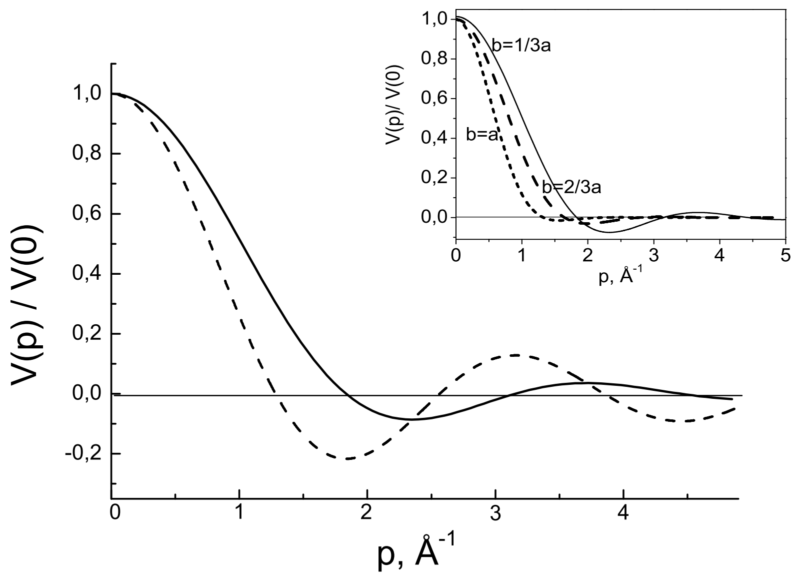


Fig. 5. The momentum dependences of the Fourier component (41) of the Fermi-type potential (40) at $b = 0$ or the Lindhardt-type potential (42) (solid curve) and of the potential of the "hard spheres" model [33] $V(p) = V_0 \sin(pa)/pa$ (dashed curve). Inset is the Fourier component for the Fermi type potential with $b \neq 0$.

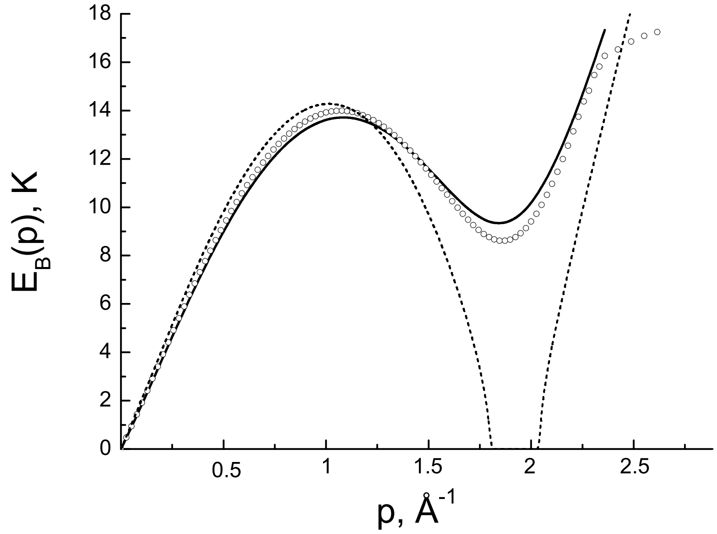


Fig. 6. The Bogolyubov quasiparticle spectrum (43) with an oscillating potential (41) (solid curve), maximally close to the experimental spectrum [25]–[29] (dotted curve) at $V_0/(4\pi a^3) = 169$ K and $a = 2.44$ Å. The dashed curve corresponds to the unstable spectrum (43) at $V_0/(4\pi a^3) = 217$ K, characterized by negative values $E_B^2(p) < 0$ in some momentum interval.

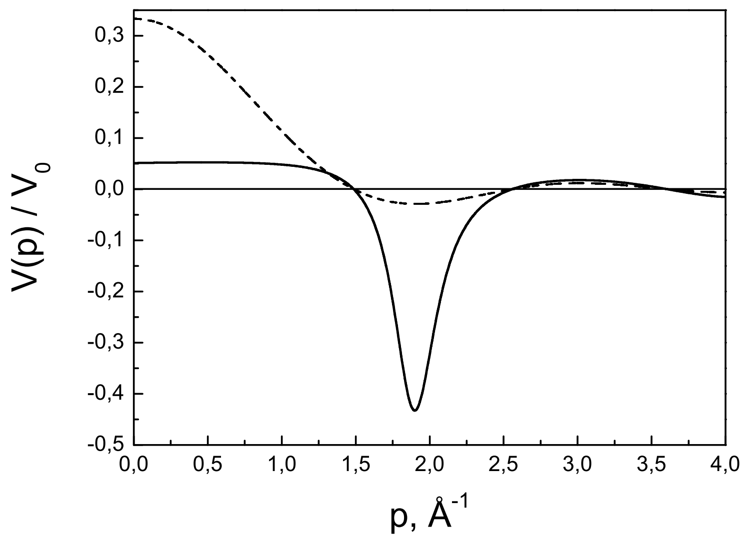


Fig. 7. The momentum dependence of the renormalized ("screened") potential (22) (solid curve) with account for the momentum dependence both of the Fourier component (41) (dashed curve) and of the polarization operator $\Pi(p, E(p))$ on the "mass shell" (see Fig. 8).

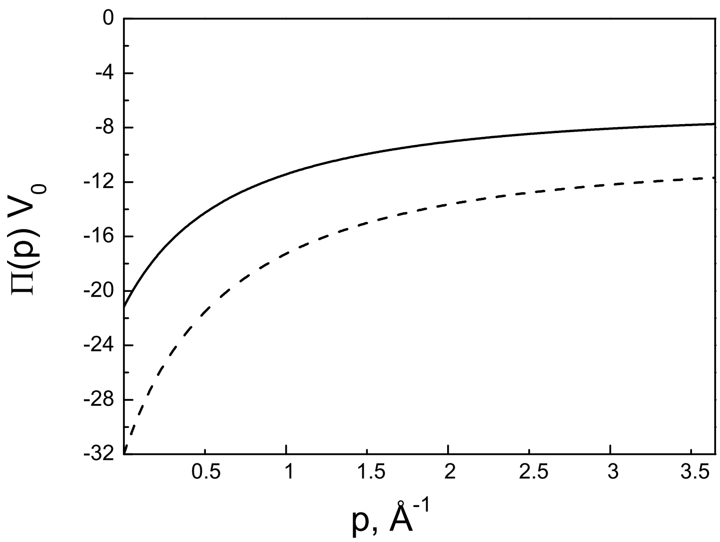
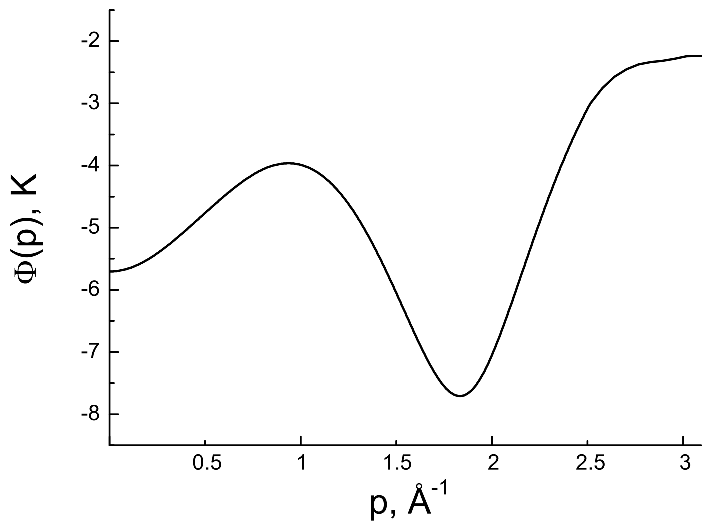
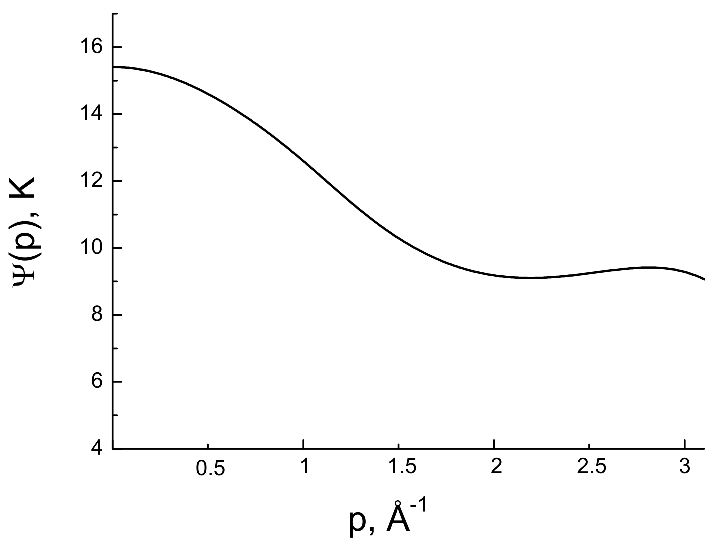


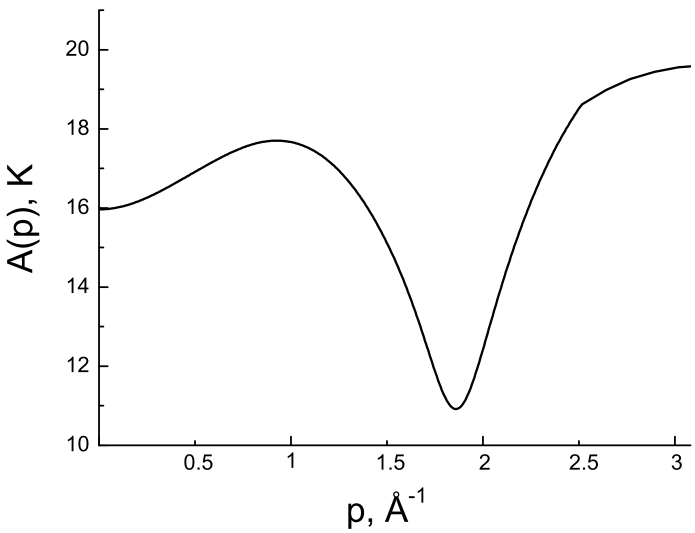
Fig. 8. The momentum dependences of the real part of the boson polarization operator on the "mass shell", $\Pi(p) \equiv \Re\,\Pi(\mathbf{p}, E(p))$, multiplied by V_0, obtained from self-consistent computations at $\Gamma = 1$ (solid curve), and of $V_0\Pi(p)\Gamma$ at $\Gamma = 1.5$ (dashed curve).



(a)



(b)



(c)

Fig. 9. The momentum dependences of the functions $\Phi(p)$ (a), $\Psi(p)$ (b), $A_0(p)$ (c), obtained from self-consistent computations at the value of the single fitting parameter $V_0/(4\pi a^3) = 1552$ K for $a = 2.44$ Å.

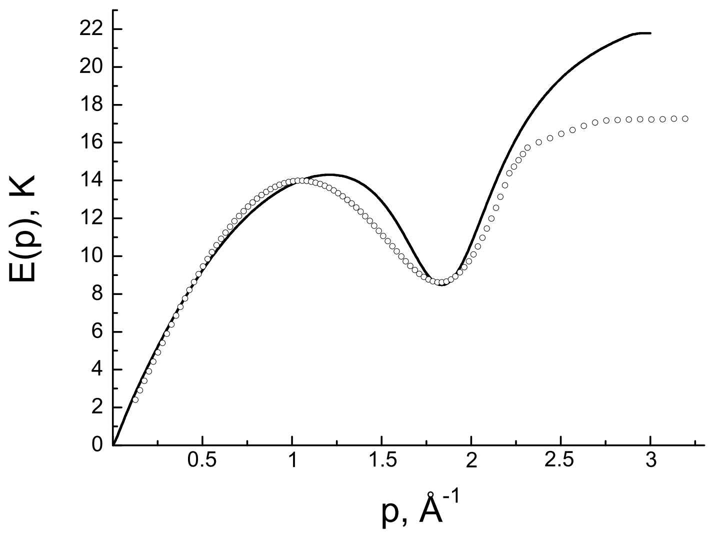


Fig. 10. The theoretical quasiparticle spectrum $E(p)$ obtained from the self-consistent computations in the model of "soft spheres" for a quantum Bose liquid vith a suppressed BEC in the approximation of constant vertex $\Lambda(p) = \Gamma(p, 0) = 1.5$ for all p (solid curve) and the empirical spectrum of elementary exitations in the SF Bose liquid ^{4}He [25]-[29] (circles).

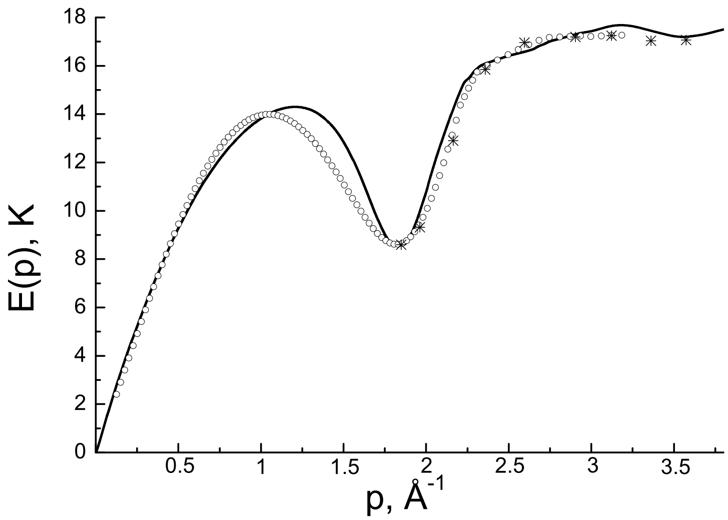


Fig. 11. The theoretical quasiparticle spectrum $E(p)$ calculated with the vertex being a decreasing function falling off from $\Gamma = 1.5$ to $\Gamma = 1.1$ in the region 2.1 Å $< p <$ 3.8 Å(solid curve) and the empirical spectra [25]-[29] (circles) and [30] (asterisks).